THE CHAIN RULE

The Chain Rule is used to differentiate Composite Functions or Function of a Function.

$$y = f\big(g(x)\big) = fog$$

Here $g(x)$ is within $f(x)$, so function *f* is composed of the function g.

Definition: Let *g* be a differential function on an interval *I*, let the range of *g* be the subset of the interval *J*, and let *f* be a differential function on *J*. Then

$$\boldsymbol{y' = f'\big(g(x)\big).\, g'(x)}$$

It is the product of the derivative of the outer function evaluated at the inner functions and the derivative of the inner function.

$\frac{dy}{dx}$ = (derivative of outside function)·(derivative of the inside function)

When using this rule it is important to

1. Identify the outer function and the inner functions.
2. Derive the outer function while leaving the inner function as is.
3. Derive the inner function.
4. Find the product of the two.

The Chain Rule is very versatile and can be used to find the derivative of more complex functions. Some of them are included here:

Generalized Power Rule is derived from the Chain Rule.

Definition: Let *g(x)* be a differentiable function and let $n \neq 0$ be an integer. Then

$$\frac{d}{dx}\left(g(x)^n\right) = n \cdot \big(g(x)\big)^{n-1} \cdot g'(x)$$

Example [3]:

$$y = (4x+3)^{10}$$

1. Identify the outer and inner functions and
2. State the derivative of these functions

Outer Function $f(x) = x^{10}$, $\quad f'(x) = 10x^9$

Inner Function $g(x) = (4x+3)$, $g'(x) = 4$

3. Find the product of the outer and inner functions

$$\frac{dy}{dx} = \frac{d}{dx}(4x+3)^{10} \cdot \frac{d}{dx}(4x+3)$$
$$= (10(4x+3)^9) \cdot (4) = 40(4x+3)^9$$

Using the Chain Rule to find a Tangent Line

Let the function f(x) be defined as y $= sinx^2$. Find the equation of the line tangent to the graph of f at $x = 1$. The equation of a tangent line at a point $x = c$ is

$$y = f'(c)(x-c) + f(c)$$

The tangent line goes through the point (1, $f(1)$).

$$f(1) = \sin(1^2) = 0.8415$$

Outer Function $f(x) = sinx$, $f'(x) = cosx$

Inner Function $g(x) = x^2$, $g'(x) = 2x$

Using Chain Rule:

$$y' = cosx^2 \cdot 2x$$
$$f'(1) = cos(1)^2 \cdot 2(1) = (0.54)2 = 1.08$$

The equation of the Tangent Line is

$$y = 1.08(x-1) + 0.8415$$

The Chain Rule can be used with Product and Quotient Rule and it can be used Multiple Times.

Example [1]: $f(x) = \frac{\sin(4x+1)}{(5x-9)^3}$

$$f'(x) = \frac{4\cos(4x+1)(5x-9) - 15\sin(4x+1)}{(5x-9)^4}$$

Some Examples [1] :

1. $f(x) = (\ln x + x^2)^3$

Identify the outer and inner functions, state their derivatives and find their product.

$$f'(x) = 3(\ln x + x^2)^2 \cdot \left(\frac{1}{x} + 2x\right)$$
$$= 3(\ln x + x^2)^2 \cdot \left(\frac{1+2x^2}{x}\right)$$
$$= \frac{3(1+2x^2)(\ln x + x^2)^2}{x}$$

$$f'(x) = \frac{(3+6x^2)\cdot(\ln(x)+x^2)^2}{x}$$

2. $f(x) = (x^2 + x)^5(3x^4 + 2x)^3$

Identify the outer and inner functions, state their derivatives and find their product.

$$f'(x) = 5(x^2+x)^4(2x+1)(3x^4+2x)^3 + 3(3x^4+2x)^2(12x^3+2)(x^2+x)^5$$
$$= (10x+5)(x^2+x)^4(3x^4+2x)^3 + (36x^3+6)(x^2+x)^5(3x^4+2x)^2$$

$$f'(x) = (x^2+x)^4(3x^4+2x)^2[(10x+5)(3x^4+2x) + (36x^3+6)(x^2+x)]$$

[1] Hartman, Siemers, Heinold, Chalishajar (2018). *APEX Calculus*. Version 4.0

[2]*2.5, MAT131 Notes Fall 2024*. (n.d.). https://www.mathnstuff.com/math/calc/131/31.M.09.30.htm

[3] https://www.mathnstuff.com/papers/condots/conpdf/dercondots.pdf